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Quantification in Ordinary Language and Proof Theory

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Résumé : Cet article dresse un rapide panorama de l'approche usuelle de la quantification, qu'elle soit ou non généralisée, en linguistique formelle et en philosophie du langage. Nous montrons que le cadre général courant va parfois à l'encontre des données linguistiques, et nous donnons quelques indications pour une approche différente basée sur la théorie de la démonstration qui, sur bien des points, s'avère plus proche de la langue que les approches les plus répandues. Nous soulignons l'importance des opérateurs tau et epsilon de Hilbert qui rendent compte respectivement de la quantification universelle et existentielle. En effet, ces opérateurs permettent de construire des représentations sémantiques en suivant la structure de la langue avec, en particulier, des groupes nominaux quantifiés qui soient des termes individuels. Nous donnons aussi des principes pour définir des règles de déduction qui correspondent aux quantificateurs généralisés.

Abstract: This paper gives an overview of the common approach to quantification and generalised quantification in formal linguistics and philosophy of language. We point out how this usual general framework represents a departure from empirical linguistic data. We briefly sketch a different idea for proof theory which is closer to the language itself than standard approaches in

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many aspects. We stress the importance of Hilbert's operators—the epsilon-operator for existential and tau-operator for universal quantifications. Indeed, these operators are helpful in the construction of a semantic representation which is close to natural language in particular with quantified noun phrases as individual terms. We also define guidelines for the design of proof rules corresponding to generalized quantifiers.

1 Foreword: empirical data on quantification

Despite the extensive study of quantification from many different viewpoints in logic, linguistics and philosophy, we think that the dominant model theoretic approach partially excludes some relevant facts that commonly occur in linguistics [Gabbay, Shehtman *et al.* 2009], [Peters & Westerståhl 2006], [Steedman 2012], [Szabolcsi 2010].

1.1 Quantification in ordinary language: common and complex

Quantifiers are quite common in ordinary language:

(1) Something happened to me yesterday.[1]

(2) A man walked into a bar.

(3) There are infinitely many primes.

(4) Every natural number can be represented as the sum of four integer squares.

(5) That way we're all writing now.

(6) a. All children are artists.

b. The children didn't miss a swimming pool, they were body boarding nearly all day.

(7) There are several reasons why few students read newspapers in the present.

(8) Two thirds of the world's inhabitants are clustered in these four regions.

1. All examples were found on the web: news, song lyrics, blogs, etc. Otherwise, i.e., to exemplify something that would not be said, we appended the superscript $^{(\mathrm{us})}$ at the end of the example.

(9) For the first time, a majority of Californians age 55 and older think that marijuana should be made legal (52% legal, 45% not legal).

(10) Many students may find it helpful to be doing practical work that contributes to what they are studying.

(11) It's fair to say that most Stones fans love Jigsaw.

(12) Most students are bored by history courses as they are usually taught.

(13) The Brits love France.

(14) MPs rejected amendments to the Consumer Rights Bill yesterday (12 January) that would have made secondary ticketing websites more transparent.

Nevertheless, their linguistic and logical study is quite complex for at least two reasons. Firstly, as we shall see, besides the well-known quantifiers used in mathematics *for all* $\forall$ (examples 6, 5, 4) and *there exists* $\exists$ (ex. 3, 2, 1), natural language makes use of a variety of quantifiers like "two thirds" (ex. 8), "few" (ex. 7), "a majority of" (ex. 9), "most" (ex. 12,11), "many" (ex. 10). Secondly, the wording of these quantifiers is often ambiguous and sometimes implicit. For instance, plural noun phrases both definite and indefinite plural articles (which have an empty realisation in English) can be used to mean "all" or "most" as can be observed in examples 13 and 14.[2]

We will now give a brief description of some (linguistically oriented) features concerning the two "mathematical" quantifiers *for all* and *there exists*.

1.2 Existential quantification in natural language

The existential quantifier is omnipresent in natural language. As soon as one speaks about something or someone, this new discourse referent is existentially quantified and can be referred to by pronouns or definite noun phrases in anaphoric chains.

(15) A man walked into a bar. His work boots were muddy, and he had dirt caked under his fingernails. He wore a grimy hoodie, and kept an arm at his waist, [...]

This sort of existential quantification is an important part of the structure of a discourse. Observe that the scope of the existential quantifier extends from one sentence (proposition) to the next one—this is one of the reasons why dynamic logics have been introduced. The behaviour of existential quantifiers actually leads to a formalism structured on existential quantification, namely

2. When meaning "all" or "most", bare plurals and definite noun phrases are possible depending on the context. The presence of "the" is probably related to the existence of possible alternatives, but this observation deserves further linguistic discussion (thanks to Gilles Zémor for helpful discussions and English expertise).

Discourse Representation Theory, in which logical formulas are built in order to better fit the linear progression of sentences. DRT has often been contrasted to compositional semantics, since DRT proceeds in a top-down direction, from large units (complex phrases) to the smallest ones (i.e., words), but by now, there exist compositional formulations of DRT. In short, existential quantifiers form a fundamental part of discourses and conversations, contributing to their structure and coherence [Kamp & Reyle 1993].

1.3 Universal quantification over potentially infinite sets in natural language

The universal quantifier, while widely used in mathematics, is rather rare in ordinary conversation, as suggested by the corpus study above. An exception is provided by quantifiers which refer to a group that can be understood from the context, as in example 5. The negative formulation by the use of *no one, nothing*, etc. is more frequent, thus *no one came* means that *every one did not come.*

In speaking about natural language it is customary to assume that quantification refers to a finite number of entities.[3] This is not always the case. We certainly meet such expressions in mathematical discussions, but also in a variety of other cases:

(16) *He wrapped up by explaining the dark future for the Universe when all the stars go away.*

(17) *All atoms are made from the same bits, which are called subatomic particles.*

(18) *Just about all sentences in the English language fall into ten patterns determined by the presence and functions of nouns, verbs, adjectives, and adverbs.*

(19) *All ideas are welcome.*

1.4 Vague quantifiers in natural language

In 1.1 we stressed that *most* provides a form of quantification. This is quite common and it also applies to infinite collections of objects:

(20) *In basic math, we're taught multiplication tables. We learn that most numbers are the answer to at least two different multiplication problems, some numbers are the answer to several, and then...*

3. It may be even more complicated, since a quantifier may refer to entities that are uncountable and continuous like the mass noun *wine* in *Peter drank up all the wine last night.*

(21) *Any module of known β is weak. Most numbers have even β and most of them are not antisymmetric.*

(22) *The number one reason why most people fail at dieting is simple: they can't stick to it.*

(23) *Since most numbers are not prime, it would waste time to check every number.*

(24) *... thus, in the limit most numbers are not prime.*

1.5 Human processing of quantification

Humans have difficulties in processing quantifiers, in particular when they occur in a phrase with an alternation of quantifiers. There are experiments in support of this viewpoint, see [Szymanik & Zajenkowski 2010]. On the other hand it is possible, even if quite rare, to find Henkin quantifiers[4] in natural language, as in *Every member of the lab knows a member of every village sports club.* What do people mean when using such quantifiers that would require a lifetime verification? The question remains unanswered. We know that our study, as well as others, via a confrontation of the formal model with what we hear or with psycholinguistic experiments may give some hints regarding the complexity of the human processing.

These few introductory words should have convinced the reader that quantification is an important and common phenomenon in natural language. Although many aspects of quantification still deserve a proper mathematical study (generalised quantifiers like “two thirds”, vague quantifiers like “a large part of”, higher order quantifiers like in “he believes whatever he is being told”, or quantification with generic elements like Hilbert's tau and epsilon) one cannot say that there is intensive mathematical research on quantification. So we agree with semanticists: quantification is an important issue which connects logic, linguistics and philosophy.

2 Remarks on the two mathematical quantifiers, *there exists* and *for all*

The two quantifiers that belong to mathematics, namely $\forall$ and $\exists$, have a proper logical description, both syntactical, with rules (since Aristotle), and semantical, with interpretations in models (since the beginning of the 20th century).

4. They are also known as branching quantifiers and they produce, if applied to a formula, its second order Skolemisation [Henkin 1961].

2.1 First order logic and completeness

An important result in first order logic is the completeness theorem, established by Gödel in 1930. The theorem shows the equivalences of two notions. It states that a formula is true in every model (1) if and only if it is provable (2)—the direction from (2) to (1) is known as the soundness theorem and it is easier to prove. The usual proof is by induction on the derivation. Other famous theorems are the compactness and Löwenheim-Skolem theorems. The former says that if every finite subset of a set F of formulae is contradiction free (admits a model), so is F, and the latter says that a theory with an infinite model and a countable language has models of every infinite cardinality [Kneale & Kneale 1986].

Those results are well known—especially completeness—and, by now, the proof theoretical study of quantification is not much developed. One should be aware that the correspondence established by the completeness theorem works only for first-order classical logic—and not for higher order logic. For instance, when second or higher order quantification is used, sub-Boolean algebras have to be considered. Moreover, when other logics are considered, more articulated structures are required to maintain that correspondence. Even in the well-studied case of intuitionistic logic, more complicated models are needed, e.g., (pre)sheaves of L-structures or Kripke models [Gabbay, Shehtman *et al.* 2009].

2.2 Some subtle differences: each/ every

Even if we only consider first order quantification, the correspondence between provability and truth is not as transparent as one hopes. Natural language provides good insights into the distinction between the proof theoretical and the model theoretical approach.

More formally, completeness expresses the equivalence between

- provability $\vdash \forall x.\ P(x)$ inferred from a proof without a free hypothesis involving a free x (a proof with a *generic* x; the precise rule is given thereafter);
- model theoretic truth, that is, in each model M_i with domain D_i, $P(x)$ holds for any $x \in D_i$, i.e., the conjunction $\&_{x \in D_i} P(x)$ holds.

An important remark is that the proof theoretical viewpoint is finite, while the model theoretic view involves two sets, which may be infinite: domains may be infinite.[5]

5. In first order logic, one can build a theory with models having a cardinality less than a given natural number n by adding to the theory the first order formula $\forall x_0 \cdots \forall x_n \forall x_n (x_n = x_0 \vee x_n = x_1 \cdots \vee x_n = x_{n-1})$. However, there is no way to build a theory with finite models. Indeed, this would require second order logic, and to use Dedekind finiteness: every functional binary predicate which is injective, is also surjective—for a given binary predicate, being a function, being surjective and being

The proof theoretic approach uses a generic element. To prove something that holds for all numbers, the proof starts with *Let n be a number...*, which means that n has no other property than being a number—this method goes back at least to Pythagoras (580–490 BC) long before the advent of logical formalisation.

It should be observed that natural language makes a similar distinction between the two notions of quantification above: it seems that *each* and *all* rather concern the complete enumeration of the elements in a collection while *every*, *any* or bare plurals (e.g., *Ducks lay eggs*) rather concern generic elements, laws and universal rules. The authors of this paper have conducted a specific study of this difference in French, comparing *chaque* ($\sim$ each?) with *tous les* ($\sim$ all ?) involving psycholinguistic experiments to support such claims [Capelier-Mourguy, Blache *et al.* 2015].

2.3 Domain restriction

Quantification in natural language is commonly formulated with a *restriction* to a given class, as "stars" in the following example:

(25) All the stars go away.

and this is also common in mathematics, despite the fact that in the standard set theoretic framework everything is a set:

$$\forall x \in \mathbb{N}, \exists u_1 \in \mathbb{N}, \exists u_2 \in \mathbb{N}, \exists u_3 \in \mathbb{N}, \exists u_4 \in \mathbb{N},\ x = (u_1)^2 + (u_2)^2 + (u_3)^2 + (u_4)^2.$$

The previous restrictions are related to a well-identified class of (inanimate) things or human beings, but there are quantifiers that are not restricted to specific classes, like *everything and something*, and these apply to a single property.

(26) *Everything has changed.*

(27) *Something happened to me yesterday.*

Using another wording for quantifiers like *all, every, some, etc.*, it is necessary to specify the class of entities (example 25) which the quantifier applies to, whereas it is not necessary for quantifiers like *everything* or *everyone*—while this later quantifier implicitly includes a restriction class, namely the class of human beings.

The restriction to a given class disappeared from mathematical logic (but not from type theory) because of the work of Frege, who insisted, for philosophical reasons, in having a single one-sorted universe. Thus, using an implicit correspondence between a set M and a predicate $M(x)$, one is able to represent quantification with restriction to a given class:

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injective are easily formulated in first order logic, but then one needs to quantify over binary predicates, see, e.g., [van Dalen 2013, chap. 5], and the family of all models may be infinite as well.

$$\forall x \in M \; P(x) \quad \equiv \forall x \; (M(x) \Rightarrow P(x))$$

$$\exists x \in M \; P(x) \quad \equiv \exists x \; (M(x) \& P(x)).$$

2.4 Natural language examples of the formulation of quantification

To present examples from the most studied quantifiers, we follow Aristotle's Square of oppositions:

- A: All As are Bs.
- I: Some As are Bs.
- E: No As are Bs.
- O: Not all As are Bs.
  (modern formulation, with a different focus: some As are not Bs)

In the original formulation, the expression A and B denoted *terms*, which are much vaguer than properties or predicates—in the Middle Ages, the theory of *suppositiones* already contributed to the clarification of the notion of *term* neater.

2.4.1 Existential affirmatives

Existential quantification (Some As are Bs) is formulated with *some*, *there is*, *a* it can also be formulated without an explicit restriction to a class of individuals by the words *someone*, or *something*. The last two sentences 30a and 30c show that natural language is finer grained: indeed natural language makes a distinction between two statements with equivalent logical forms (resp. 30b and 30d): the difference lies in the *focus* of the sentence, which accounts for the absence of t sentences like 30c from corpora, because people are not interested in *crooks* as a class.

(28) *There's a tramp sittin' on my doorstep.*

(29) *Some girls give me money.*

(30) *Something happened to me yesterday.*

a. *Some politicians are crooks.*

b. $\exists x. \quad politician(x) \& crook(x).$

c. *Some crooks are politicians.*

d. $\exists x. \quad crook(x) \& politician(x).$

2.4.2 Existential negatives

Observe that the negative existential (the fourth corner of Aristotle's Square of oppositions) is NOT lexicalised (in English and other languages). Psycholinguists know that this is harder to grasp and rather ambiguous. Moreover, in ordinary conversation, an negative existential statement is often misunderstood as a universal negative statement (e.g., example 31 may be understood as 32). For a psycholinguistic study of the human understanding of such sentences, see e.g., [Delfitto & Vender 2010] where more references can be found. Also observe that in our unambiguous rephrasing 35 of 34, the focus has changed from the individuals who can be funded to those who cannot, making the rephrased statement more difficult to understand.

(31) *Not every picture tells a story.*

(32) *No picture tells a story*$^{(\mathrm{us})}$.

(33) *Some students do not participate in group experiments or projects.*

(34) *Everyone is entitled to an opinion, but not every opinion is entitled to student government funding.*

(35) *Everyone is entitled to an opinion, but some opinions are not entitled to student government funding.*$^{(\mathrm{us})}$

2.4.3 Universal affirmatives, especially on infinite domains

We have already provided some examples of universal statements. Here are some more, ranging on a potentially infinite domain:

(36) *Each star in the sky is an enormous glowing ball of gas.*

(37) *All groups of stars are held together by gravitational forces.*

(38) *Terence Tao, a Fields medalist, has published a paper that proves that every odd number greater than 1 is the sum of at most five primes.*

2.4.4 Universal negatives, especially on infinite domains

The universal negative statements are expressed either by *no* or, without a restriction to a class, by *no one*, *nothing.* Here are some examples:

(39) *Because no planet's orbit is perfectly circular, the distance of each varies over the course of its year.*

(40) *Nothing's gonna change my world.*

(41) *Porterfield went where no colleague had gone previously this season, realising three figures.*

2.5 Proof theory of mathematical quantifiers $\exists$ and $\forall$

How do $\exists$ and $\forall$ work from a proof theoretical viewpoint? They both have introduction and elimination rules:

- Universal quantifier
  - The $\forall$ introduction rule says, as above, that when a property has been established for an x which does not enjoy any particular property (i.e., is not free in any hypothesis), one can conclude that the property holds for all individuals:

$$\dfrac{\begin{array}{c}\text{no free occurrence of } x\\ \text{in any term of any } H_i\\ H_1, \ldots, H_n \vdash P(x)\end{array}}{H_1, \ldots, H_n \vdash \forall x.\ P(x)}\ \forall_i$$

  - The $\forall$ elimination rule says that when a property has been established for all individuals it can be inferred for any particular individual (or term):[6]

$$\dfrac{H_1, \ldots, H_n \vdash \forall x.\ P(x)}{H_1, \ldots, H_n \vdash P(a)}\ \forall_e$$

- Existential quantifier
  - The $\exists$ introduction rule says that whenever a property P holds for an individual (or a term) a one can infer that there exists an individual enjoying P

$$\dfrac{H_1, \ldots, H_n \vdash P(a)}{H_1, \ldots, H_n \vdash \exists x.\ P(x)}\ \exists_i$$

  - The $\exists$ elimination rule is more complicated. It says that if assuming $P(x)$ and nothing more about x, we derive C, which does not depend on x, and if we have a proof of $\exists x.\ P(x)$, we can conclude C without the hypothesis $P(x)$:

$$\dfrac{H'_1, \ldots, H'_p \vdash \exists x.\ P(x) \qquad \begin{array}{c}\text{no free occurrence of } x\\ \text{in any term of any } H_i\\ \text{nor in any term of } C\\ P(x), H_1, \ldots, H_n \vdash C\end{array}}{H'_1, \ldots, H'_p, H_1, \ldots, H_n \vdash C}\ \exists_e$$

6. The addition of terms to first order logic does not change the expressive power, since terms can be defined from predicates: a function f is a binary predicate $F(\_,\_)$ such that $\forall x \exists y F(x,y)$ and $\forall x \forall y \forall z (F(x,v) \& F(x,z) \Rightarrow y = z)$.

this rule could also be expressed by a rule stating that from the hypothesis $\exists x.P(x)$ it is possible to conclude $P(x)$, where x does not occur in the other hypotheses and does not occur in the conclusion of the proof.

2.6 Individual concepts and second order

An alternative way to deal with quantification is to use individual concepts, i.e., properties that hold for exactly one individual. This is the reason why in the Discourse Representation Theory some people write $John(x)$ while others write **j**: the former notation describes an individual concept $John(\_)$ which says that the property of being John is true for exactly one individual. This can be formulated in second and first order logic: X is an individual concept whenever $C(X) = (\forall x \forall y (X(x) \wedge X(y) \Rightarrow x = y)) \wedge \exists x.\ X(x)$ holds—there are variants in which the existence is not required. Universal quantification can be phrased with individual concepts via second order quantification. Indeed, $\forall x.\ P(x)$ can be expressed as $\forall X (C(X) \Rightarrow P^{\#}(X))$ with $P^{\#}(X) = \exists x.\ (X(x) \wedge P(x))$. Conversely, given a property Q of individual concepts, the quantification $\forall X.\ (C(X) \Rightarrow Q(X))$ corresponds to $\forall x.\ Q^{b}(x)$ with $Q^{b}(x) = \exists X.\ (C(X) \wedge X(x) \wedge Q(X))$. The equivalences are established by formal proof in second order logic using second order quantification, and by duality the same result holds for existential quantification. There is a variant according to which non emptiness is not required, since it is possible to name individuals that do not exist. When this is the case, one form of quantification is stronger than the other. One might wonder why one uses individual concepts and therefore second order logic. One reason is that individual concepts are closer to the notion of term in Ancient and Medieval logic: both *Socrates* and *human beings* are *terms*. A more concrete reason is that, in possible worlds semantics with rigid designators (the standard interpretation in formal semantics), it is impossible to interpret *I do not believe that Tully is Cicero* [Lacroix 2011].

2.7 Proof and refutation semantics

Proof theoretical approaches naturally lead to rules of refutation. How does one refute a universal statement $\forall x P(x)$? Of course, this amounts to proving its negation, the existential statement $\exists x \neg P(x)$, but there are different ways to do so. Consider the following example:

(42) *[The AKC notes] that any dog may bite.*

(43) *No, Rex would never bite.*(us)

(44) *Basset hounds do not bite.*(us)

The difference is that the first answer picks up an element in the relevant model, while the second answer remains with generic elements. This is related to the Avicenian idea that a property of a term (individual or not) is always asserted for the term as part of a class: the view of Avicenna is more related to type theory than to the Fregean view of a single universe.

3 Hilbert's operators

Russell introduced the operator[7] ι for definite descriptions, meaning *the one and unique individual such that P.*

Hilbert used *generic* elements intensively for quantification (with Ackerman and Bernays). This study was initiated by Hilbert in [Hilbert 1922] and it culminated in the second volume of *Grundlagen der Mathematik* with Bernays [Hilbert & Bernays 1939]. It should be stressed that the presentation of these operators by Hilbert relied on natural language examples—an uncommon feature in Hilbert's writings. Hilbert's ϵ operator has recently led to important work in linguistics, in particular with von Heusinger's work [Egli & Heusinger 1995], [Heusinger 1997, 2004].

3.1 An ancestor to Hilbert operators: Russell's operator for definite descriptions

The first step, due to Russell, was to denote by $\iota_x.\ F(x)$ the unique individual enjoying the property F in a definite description, as in the first example in the list below and in *Principia Mathematica.* When existence or unicity fails, it is said that the reference of $\iota_x.\ F(x)$ is the null class. A predicate applied to such a non existing individual should be false. This results in a complicated distinction between primary and secondary occurrences of a definite noun phrase to avoid that the two first sentences, which are the negation one of another, are both false [Russell 1905].

(45) The present king of France is bald.

(46) The present king of France is not bald.

(47) The present president of France *was born in Rouen.*$^{(\mathrm{us})}$
(existence and uniqueness hold)

(48) The present king of France *was born in Pau.*$^{(\mathrm{us})}$
(existence fails)

(49) The minister *was born in Le Mans.*$^{(\mathrm{us})}$
(uniqueness fails)

7. Curry suggested in [Curry & Feys 1958, 275] to call a *subnector* an operator applying to a formula with a free variable, binding this free variable and yielding an individual term. The term is precise and well chosen but we shall not use it, because only a few people do.

3.2 Hilbert's operator ϵ-operator and τ-operator

Hilbert introduced for each formula an ϵ-term, i.e., an existential term containing the ϵ-operator: given a formula $F(x)$, with a free variable x, one defines the term $\epsilon_x.\ F$, where all the occurrences of x in F are bound (this is the original notation, nowadays this term is often written as $\epsilon_x.\ F(x)$). The ϵ-axiom states that whenever an element, say a, enjoys F, then the ϵ-term $\epsilon_x.\ F$ enjoys F.

Dually, Hilbert introduced for each formula F a τ-term $\tau_x.\ F(x)$, i.e., a universal term containing the τ-operator, which corresponds to the generic elements used in mathematical proofs. The τ-axiom states that if $\tau_x.\ F(x)$ enjoys the property F then every individual does.

By the ϵ-axiom, $\epsilon_x.\neg F(x)$ plays the role of $\tau_x.\ F(x)$; conversely by τ-axiom, $\tau_x.\ \neg F(x)$ plays the role of $\epsilon_x.\ F(x)$.

The evident deduction rules for τ and ϵ are as follows:

- Introduction rule for ϵ-terms: from $A(c)$ infer $A(\epsilon_x.\ A(x))$.
- Elimination rule for ϵ-terms: from $A(\epsilon_x.\ A(x))$ infer $A(x)$, where no free occurrences of x are in the context of the proof.
- Introduction rule for τ-terms: from $A(x)$ infer $A(\tau_x.\ A(x))$, provided that no free occurrences of x are in the context of the proof.
- Elimination rule for τ-terms: from $A(\tau_x.\ A(x))$ infer $A(c)$.

These rules show that $F(\tau_x.\ F(x)) \equiv \forall x.\ F(x)$ and $F(\epsilon_x.\ F(x)) \equiv \exists x.\ F(x)$. Thus existential and universal quantifiers may be defined by means of Hilbert's operators and, by duality, one of these is redundant. We keep, as it is customary, the ϵ-operator and thus the logic with epsilon operator is known as the epsilon calculus.

Hilbert turned these symbols into an attractive mathematical object, since it allows to fully describe quantification with simple rules. The first and second epsilon theorem basically say that the epsilon calculus is at least as expressive as first order logic.

First ϵ-theorem When inferring a formula C with neither ϵ-symbol nor quantifier from formulas Γ not involving the ϵ-symbol nor quantifiers the derivation can be done within quantifier free predicate calculus.

Second ϵ-theorem When inferring a formula C without ϵ-symbol from formulae Γ not involving the ϵ-symbol the derivation can be done within predicate calculus.

3.3 Expressive power of Hilbert's epsilon

Indeed the epsilon calculus is more expressive than first order logic, since in the former there are formulas which are not equivalent to any first order formula, e.g., formulas of the form $P(\epsilon_x.\ F(x))$, where P and F are two nonequivalent formulas. Nevertheless, it is remarkable to note that every formula of the form $P(\epsilon_x.\ F(x))$ "stands between" two first order formulas, such as in the following:

$$\exists x\ F(x) \wedge \forall y\ (F(y) \Rightarrow P(y)) \vdash P(\epsilon_x.\ F(x)) \quad \text{and} \quad P(\epsilon_x.\ F(x)) \vdash \exists x\ P(x).$$

The first and second epsilon theorem provided the first correct proof of Herbrand's theorem (long before mistakes where found and solved by Goldfarb) and a proof of the consistency of first order Peano arithmetic (contemporary to Gentzen's proof). The extension of the cut elimination patterns to these operators does not seem too complicated and this work as already been undertaken by [Mints 2008].

3.4 Individual concept and Hilbert's epsilon

Recall that in first order logic an individual concept is a formula P such that the two following formulas hold:

$$\forall x\ \forall y\ ((P(x) \wedge P(y)) \rightarrow x = y) \qquad \text{and} \qquad \exists x\ P(x).$$

A straightforward example of an individual concept is provided by a formula of the form $x = t$ where t is a term.

Using the same definition in Hilbert's epsilon calculus, we can show that there are no individual concepts other than equalities, i.e., if P is an individual concept, then

$$P(x) \dashv\vdash x = \epsilon_x.\ P(x)$$

In fact, if P is an individual concept

$$\dfrac{\dfrac{P(x) \qquad \dfrac{\exists x\ P(x)}{P(\epsilon_x.\ P(x))}}{P(x) \wedge P(\epsilon_x.\ P(x))}}{x = \epsilon_x.\ P(x)} \qquad\qquad \dfrac{\dfrac{x = \epsilon_x.\ P(x) \qquad \dfrac{\exists x\ P(x)}{P(\epsilon_x.\ P(x))}}{x = \epsilon_x.\ P(x) \wedge P(\epsilon_x.\ P(x))}}{P(x)}$$

3.5 Hilbert's operators in natural language

In Hilbert's book, these operators are explained with natural language examples, but a very important linguistic property is not stated. The $\epsilon_x.\ F(x)$ has a type (both in the intuitive and in the formal sense) of a noun phrase, and is meant to be the argument of a predicate (for instance the subject of a verb), thus being a *suppositio* in the medieval sense [de Libera 1996], [Kneale & Kneale 1986].

Nowadays, we witness a renewed interest in these epsilon formulations of quantification, in particular by von Heusinger [Heusinger 1997, 2004]. He actually uses two variants of the epsilon, one ϵ^a corresponding to the indefinite article "a" and the other one ϵ^d corresponding to definite determiners like "the" or "this". When interpreting an indefinite noun phrase like "a man", the corresponding term $\epsilon^a x.man(x)$ is to be interpreted as a "new" man. When interpreting a definite noun phrase like "the man" the corresponding term $\epsilon^d x.man(x)$, where uniqueness constraints are left out, is to be interpreted as the most salient man in the possible discourse referents, leaving out the uniqueness of the iota operator of Russell. It is not clear whether the equivalence with ordinary existential quantification is retained. Von Heusinger constructs an epsilon term whenever there is an expression like *a man* or *the man*, but it is not clear how one can assert that $man(\epsilon_x.\ man(x))$. Another difficulty is that different occurrences of the very same logical term $\epsilon^a x.man(x)$ ought to have different interpretations—and the various occurrences of $\epsilon^d x.man(x)$ may have different interpretation as well.[8]

4 Generalised quantifiers

The abundant literature on generalised quantifiers follows and enriches Frege's view with model theory. Initiated with [Mostowski 1957], from [Lindström 1966] it is assumed that generalized quantifiers like *thirty per cent* or *many* or *most* are functions of two predicates viewed as sets and this view was successfully developed in [Barwise & Cooper 1981] for natural language semantics.

Functions of two predicates are required, since the aforementioned Fregean way to handle restricted quantification in a one-sorted logic ($\forall x \in M\ P(x) \quad \equiv \forall x\ (M(x) \Rightarrow P(x))$) does not apply. Indeed, as an easy exercise shows, the two sentences below are not equivalent:

(50) Most students go out on Thursday nights.

(51) For most individuals, if they are students then they go out on Thursday nights.

Generalised quantifiers, viewed as functions of two predicates or subsets of the domain, are classified according to their behaviour with respect to their two arguments: covariant, contravariant, and properties with respect to existential quantification (weakness). Some rules are provided, but *in fine* they amount

8. The notation of von Heusinger may be misleading. He uses η for an existential term to be interpreted by a new element, commonly introduced by "a". He uses ϵ for interpreting the definite article, from which he leaves out the unicity condition, and this term is interpreted as the most salient element. In Hilbert one also finds a η operator: the difference with Hilbert's ϵ-term is that η terms can only be introduced in the language when the existence is certain. Hilbert does not worry about the model theoretic interpretation of these terms [Hilbert & Bernays 1939].

to count how many elements are in the class corresponding to the restriction, in the main predicate itself and in their intersections. This approach leaves out sets that are infinite or potentially infinite. The proof rules that are given always refer to an intended model, and therefore they cannot be considered proof rules in the ordinary sense.

This sometimes leads to inaccurate results. For instance, *most* is defined as *the majority of* although their actual usage is quite different: considering the two following sentences, one can assert the first one but not the second one:

(52) *The majority of French electors voted Hollande in 2012.*

(53) *Most French electors voted Hollande in 2012.*

Indeed, *most* is a vague quantifier and, as *many*, its validity depends on a flexible and context dependent percentage. The fact that the usual cardinality approach, the usual one, is wrong is proved by the following examples:

(54) a. *Most numbers are not prime.*

b. numbers and prime numbers have the same cardinality: $\aleph_0$.

(55) a. *Most people want to have children.*

b. which set does "people" refer to? is it a finite set?

Indeed, what is clear is that a measure is required, but counting is not the only way to measure a set. For instance, as far as cardinality is concerned, the first sentence would be false, although we agree with this statement from the introduction of an advanced book on number theory.[9]

Another problematic point is to consider generalised quantifiers as functions of *several* predicates (following [Lindström 1966]), while mathematical quantifiers are functions of a *single* predicate: given a predicate $P(x)$ one can form $\forall x.\, P(x)$. General quantifiers depend on several predicates, which usually are the main predicate and the restriction to a given class. The quantifier Q (some, all, most,...) in a sentence like Q *children sleep* is supposed, in the standard view to be a function that applies to the predicate *children* (restriction class) and to the main predicate *sleep*. Hence Q *children* cannot be interpreted in this standard view which goes against the syntactic and cognitive structure of language. Indeed, if a speaker starts a sentence with Q *children* before coming to the verb, the hearer already has in mind an image of what Q *children* is, only by the presence of the word *children*—"a child, many children, all children, two children,..." should be viewed as generic individuals or as subsets of the set of children and that's not the way they are interpreted in the standard interpretation with generalised quantifiers.

Drawbacks of the standard approach is avoided when using (generalised) Hilbert operators or choice functions for interpreting determiners [Retoré 2013,

9. We agree because the percentage of primes among the n first integers decreases and its limit is 0.

2014b]. This can be viewed as a formal development of Geach's ideas on *terms* and *predicates* [Geach 1962]. This approach with individual terms can be extended to generalised quantifiers like "most" as done in [Retoré 2012].

5 Towards rules for some generalised quantifiers

Keeping the idea of individual terms that express quantification, we started to study generalised quantification from two converging viewpoints. One viewpoint is the interface between syntax and semantics, i.e., how do we map the syntactic structure of a sentence to a logical formula that represents the semantics of that sentence. The other viewpoint is the proof theory of terms and formulas expressing generalised quantification: how do we prove and refute them?

5.1 Semantic representations with generalised quantifiers

Regarding the first viewpoint, we advocate a typed version of Hilbert's generic elements, as in [Retoré 2012]. Consider the following sentences:

(56) Most dogs bite.

(57) Most of the students that passed algebra passed logic.

In those sentences, the first restriction class, namely *dogs*, can be viewed as a type, while in the second example the restriction class *students that passed algebra* is a formula with a unique free variable. Those operators are constants of second order lambda calculus (system F), which is the typed lambda calculus that we use as a framework for natural language semantics [Retoré 2014a]. We actually have two distinct versions of the operator introducing a *most* generic element. A first one maps a type to an element in the type itself so its type is $\Pi X.X$. The second one applies to a term, a predicate P over some type U, i.e., P is of type $T \to \mathit{prop}$ (here $T = \mathit{student}$) as argument and yields an element x of this type U (here *student*) with additionally the presupposition that this x enjoys the property P (here the added presupposition is that the generic *student* has *passed algebra*). Hence the type of the type of the second most is $\Pi X.(X \to \mathit{prop}) \to X$: it first applies to a type U (here: students) and then to a predicate over U objects (*passed algebra*).[10]

We propose to do so for all generalised quantifiers, that is, to have generic elements corresponding to them, defined as typed Hilbert's operators. This

10. There are no closed terms of type $\Pi X.X$ nor of type $\Pi X.(X \to \mathit{prop}) \to X$, because these formulae cannot be proved, but adding constants with these types is safe.

is a way to compute their semantic representations, but it does not say how they are interpreted (we only provided some hints), neither how one asserts and refutes sentences with such quantifiers [Retoré 2014b].

5.2 Proof theoretical meaning of generalized quantification

Concerning the *the majority of* quantifier, we thought about what could be the rules for proofs (introduction rules) and refutations (elimination rules) for such a quantifier. There are at least two ways to refute that the majority of the A have the property P (where for the majority of we mean strictly more than 50%) :

- Only the minority of the A has the property P.
- There is another property Q which holds for the majority of the A, with no A satisfying both P and Q.

One way to add a generalised quantifier to a proof system is to introduce a pair of dual quantifiers, a variant $\forall^*$ of $\forall$ and a variant $\exists^*$ of $\exists$.

If they can be compared, one has to choose one of the following two possibilities, the first one being unlikely:

- $\forall^* x.\ A(x)$ implies $\forall x.\ A(x)$ and so $\exists x.\ A(x)$ implies $\exists^* x.\ A(x)$.
- $\exists^* x.\ A(x)$ implies $\exists x.\ A(x)$ and so $\forall x.\ A(x)$ implies $\forall^* x.\ A(x)$.

In both the cases, one of the two new quantifiers is obtained by extending an existing rule, and the other one by restricting an existing rule. Although a complete set of rules is not available, it is likely that a family of rules and refutation techniques that would model a large part of those quantifiers' behaviour can be found.

One can also say that the new quantifier(s) cannot be compared with the standard ones. For instance there is no reason to compare two variants of the universal quantifier, one say $\forall^\tau$, expressing the rule that every individual (or the generic individual with respect to P) has property P (such rules admit exceptions but apply to any situation) and the other say, $\forall^\&$, expressing the fact that by coincidence each individual has property P, and this is a conjunction over all the elements in the domain (no exception but such statements are context dependent)—hence the domain must be known for $\forall^\&$. The introduction rule for generic quantification should be the usual introduction rule: from $A(x)$ with no assumption on x infer $\forall^\tau x P(x)$. The introduction rule for quantification as conjunction should look like Gentzen's ω-rule for arithmetic [Gentzen 1936]: from $P(x)$ for any x in the domain, infer $\forall^\& x P(x)$. The elimination rule of $\forall^\& x P(x)$ should be as expected: one can recover $P(x)$ for any x in the domain. The elimination rule for the $\forall^\tau$ is complicated since you can infer $P(x)$ only for those x that are not exceptions, and handling exceptions would require a substantial extension of the logical framework.

6 Conclusion

After some criticisms of the common set theoretic approach, we gave some hints for a proof theoretical approach to quantification. One idea is to use typed versions of Hilbert operator for computing the semantic representations and we are presently doing so in a type theoretical framework [Retoré 2014a]. Another idea that we are exploring is to use principles on proofs and cut-elimination/normalisation to design the rules of generalised quantifiers, and this can also be discussed in our typed theoretical framework.

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